

NON-SELFADJOINT OPERATOR ALGEBRAS GENERATED BY WEIGHTED SHIFTS ON FOCK SPACE

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ABSTRACT. Non-commutative multi-variable versions of weighted shifts arise naturally as ‘weighted’ left creation operators acting on Fock space. We investigate the unital WOT-closed algebras they generate. The unweighted case yields non-commutative analytic Toeplitz algebras. The commutant can be described in terms of weighted right creation operators when the weights satisfy a condition specific to the non-commutative setting. We prove these algebras are reflexive when the eigenvalues for the adjoint algebra include an open set in complex n -space, and provide a new elementary proof of reflexivity for the unweighted case. We compute eigenvalues for the adjoint algebras in general, finding geometry not present in the single variable setting. Motivated by this work, we obtain general information on the spectral theory for non-commuting n -tuples of operators.

The study of non-commutative multi-variable versions of weighted shift operators was initiated in [13]. These n -tuples arise naturally as ‘weighted’ left creation operators acting on Fock space. Certain C^* -algebras determined by these *weighted shifts on Fock space* played a crucial role in [13], and the entire class is currently under investigation in [2]. In this paper, we consider non-selfadjoint algebras generated by these operators. In particular, we are interested in the weak operator topology closed non-selfadjoint algebras they generate. There is now an extensive body of literature for the unweighted case. The algebras generated by the left creation operators have been established as the appropriate *non-commutative analytic Toeplitz algebras* (for instance see [1, 5, 6, 14, 17, 18]).

Our motivation with this work is twofold: we wish to establish non-trivial analogues of results obtained for standard weighted shifts. Towards this end we are motivated by the well-known survey article [21].

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At the same time, we wish to expose differences encountered in this new non-commutative setting. Many of the weight conditions we obtain are exclusive to this setting in that they reduce to trivialities in the single variable case $n = 1$.

The first section contains a review of the basic facts for these weighted shifts, as well as an introduction to the algebras \mathfrak{L}_Λ we study and the associated weight functions. In the second section we show that, under a weight condition specific to the non-commutative setting, a commutant theorem can be proved which generalizes the single variable commutant theorem [21, 22], as well as the unweighted $n \geq 2$ case [5, 18]. The condition amounts to requiring the boundedness of particular weighted right creation operators. This theorem leads to good internal information on the algebras.

We investigate the reflexivity of \mathfrak{L}_Λ in the third section. If the set of eigenvalues for \mathfrak{L}_Λ^* includes an open set in \mathbb{C}^n , then \mathfrak{L}_Λ is reflexive. Our proof gives a new elementary proof of the reflexivity of non-commutative analytic Toeplitz algebras \mathfrak{L}_n [1, 5]. We pose some open questions related to the reflexivity of \mathfrak{L}_Λ .

In the fourth section we compute eigenvalues for \mathfrak{L}_Λ^* , finding geometry not present in other settings. In general there is a wealth of eigenvalues if there are any which are non-zero. Using this analysis as motivation, we discuss the spectral theory for non-commuting n -tuples of operators in the fifth section. In particular, we show that the natural notion of a full spectrum [23, 24] does not carry over to the non-commutative several variable case. Nonetheless, substantial information can still be obtained from one-sided spectra.

In the final section we present examples of shifts satisfying the various weight conditions derived throughout the paper. Our examples include certain subclasses of the periodic shifts introduced in [13]. We also discover other new subclasses which help illustrate points.

1. INTRODUCTION

For positive integers $n \geq 2$, let \mathbb{F}_n^+ be the unital free semigroup on n non-commuting letters $\{1, 2, \dots, n\}$. One way to realize n -variable Fock space is as the Hilbert space $\mathcal{H}_n = \ell^2(\mathbb{F}_n^+)$, where an orthonormal basis is given by vectors corresponding to words $\{\xi_w : w \in \mathbb{F}_n^+\}$ with the *vacuum vector* ξ_e corresponding to the unit or empty word e in \mathbb{F}_n^+ . A *weighted shift* on Fock space is an n -tuple $S = (S_1, \dots, S_n)$ of operators $S_i \in \mathcal{B}(\mathcal{H})$ such that there is a unitary $U : \mathcal{H}_n \rightarrow \mathcal{H}$ and operators $T_i = U^* S_i U$ for which there are scalars $\Lambda = \{\lambda_{i,w}\}$ with

$$(1) \quad T_i \xi_w = \lambda_{i,w} \xi_{iw} \quad \text{for } w \in \mathbb{F}_n^+ \quad \text{and} \quad 1 \leq i \leq n.$$

A helpful pictorial way to think of $T = (T_1, \dots, T_n)$ is as a ‘weighted Fock space tree’. This was outlined in [13], but the basic idea is the following: The vacuum vector ξ_e corresponds to the vertex lying at the top of the tree, and every other basis vector ξ_w corresponds to a vertex with exactly n edges leaving it downwards to the vertices for ξ_{iw} , and a unique edge coming into it from above. When we regard the edges as weighted by the scalars $\lambda_{i,w}$, we can think of this weighted Fock space tree as completely describing the actions of T_1, \dots, T_n .

As with standard weighted shifts we make some simplifying assumptions. For the sake of brevity, we assume that the weighted shifts $T = (T_1, \dots, T_n)$ act on \mathcal{H}_n as in (1) with weights given by $\Lambda = \{\lambda_{i,w}\}$. Further, a unitary $U \in \mathcal{B}(\mathcal{H}_n)$ which is diagonal with respect to $\{\xi_w\}$ can be constructed for which the weighted shift $(U^*T_1U, \dots, U^*T_nU)$ has the non-negative weights $\{|\lambda_{i,w}|\}$. In addition, all T_i are injective precisely when each $\lambda_{i,w} \neq 0$, since there are no sinks in the weighted tree. Hence we shall make the following assumption on $\Lambda = \{\lambda_{i,w}\}$:

$$\text{Assumption: } \lambda_{i,w} > 0 \quad \text{for } w \in \mathbb{F}_n^+ \quad \text{and } 1 \leq i \leq n.$$

We now define the algebras we wish to study.

Definition 1.1. Given a weighted shift $T = (T_1, \dots, T_n)$ on \mathcal{H}_n with weights $\Lambda = \{\lambda_{i,w}\}$, define \mathfrak{L}_Λ to be the unital WOT-closed algebra generated by the operators $\{T_1, \dots, T_n\}$.

The unweighted case $\lambda_{i,w} \equiv 1$ yields the *left creation operators* $L = (L_1, \dots, L_n)$, of theoretical physics and free probability theory. The algebras in this case are the so called non-commutative analytic Toeplitz algebras \mathfrak{L}_n , which are also the WOT-closed algebras generated by the left regular representation of \mathbb{F}_n^+ (see [1, 5, 6, 14, 17, 18]). The operators $L = (L_1, \dots, L_n)$ can be regarded as non-commutative multi-variable versions of the unilateral shift, hence considering weighted versions seems like a natural line of research. The following facts were easily derived in [13] for $T = (T_1, \dots, T_n)$:

- (i) Each $T_i = L_i W_i$ where W_i is the diagonal weight operator, which is positive and injective with our above assumption, given by $W_i \xi_w = \lambda_{i,w} \xi_w$.
- (ii) $\|T_i\| = \sup_w \{\lambda_{i,w}\}$, for $1 \leq i \leq n$, and $\|T\| = \sup_{i,w} \{\lambda_{i,w}\}$.

As in [2, 3], it is sensible to introduce certain weight functions when studying multi-variable weighted shifts.

Definition 1.2. Given weights $\Lambda = \{\lambda_{i,w}\}$, define the associated *weight function* $W : \mathbb{F}_n^+ \times \mathbb{F}_n^+ \longrightarrow \mathbb{R}_+$ by

$$W(u, w) = \begin{cases} \lambda_{i_1, u} \lambda_{i_2, i_1 u} \cdots \lambda_{i_k, i_{k-1} \cdots i_1 u} & \text{if } w = i_k \cdots i_1 \\ 1 & \text{if } w = e \end{cases}$$

For words $w = i_k \cdots i_1 \in \mathbb{F}_n^+$, we introduce the notational convenience $T_w := T_{i_k} \cdots T_{i_1}$. Then we have

$$T_w \xi_u = W(u, w) \xi_{wu} \quad \text{for } u, w \in \mathbb{F}_n^+.$$

With respect to the weighted tree structure, $W(u, w)$ is the product of all weights picked up when one moves from the vertex ξ_u to the vertex ξ_{wu} . This function satisfies a cocycle condition given by

$$(2) \quad W(u, vw) = W(wu, v)W(u, w).$$

This can be easily seen when viewing the tree structure, but we supply a proof as well.

Proposition 1.3. *The weight function $W : \mathbb{F}_n^+ \times \mathbb{F}_n^+ \rightarrow \mathbb{R}_+$ satisfies the cocycle condition (2).*

Proof. Let $u, v = i_k \cdots i_1$, and $w = j_l \cdots j_1$ be words in \mathbb{F}_n^+ . Then by definition we have

$$\begin{aligned} W(u, vw) &= W(u, (i_k \cdots i_1)(j_l \cdots j_1)) \\ &= (\lambda_{j_1, u} \cdots \lambda_{j_l, j_{l-1} \cdots j_1 u}) (\lambda_{i_1, wu} \cdots \lambda_{i_k, i_{k-1} \cdots i_1 wu}) \\ &= W(u, w)W(wu, v). \end{aligned}$$

Finally, the formula clearly holds when $v = e$ or $w = e$. ■

2. COMMUTANT AND BASIC PROPERTIES OF \mathfrak{L}_Λ

We begin by investigating the commutant structure. For $n = 1$, the operators commuting with a weighted shift are precisely the WOT-limits of polynomials in the shift; in other words, the WOT-closed algebra generated by the shift is its own commutant [21, 22]. For $n \geq 2$, the commutant of \mathfrak{L}_n is the algebra \mathfrak{R}_n determined by the right regular representation of \mathbb{F}_n^+ [5, 18]. This is the unital WOT-closed algebra generated by isometries $R_i \in \mathcal{B}(\mathcal{H}_n)$ defined by $R_i \xi_w = \xi_{wi}$. Thus the natural generalization of these results to our setting would require commutants determined by *weighted right regular representations*. As the first lemma shows, if this is to be the case, we are forced into specific choices for the corresponding weighted right creation operators.

Lemma 2.1. *Let $T = (T_1, \dots, T_n)$ be a weighted shift. If there are operators $S = (S_1, \dots, S_n)$ with $S_i \in \mathcal{B}(\mathcal{H}_n)$ which satisfy*

$$S_i \xi_w = \mu_{i,w} \xi_{wi} \quad \text{for } w \in \mathbb{F}_n^+ \quad \text{and } 1 \leq i \leq n,$$

then $T_i S_j = S_j T_i$ for $1 \leq i, j \leq n$ if and only if the scalars

$$(3) \quad \mu_{i,w} = c_i W(i, w) W(e, w)^{-1} \quad \text{for } w \in \mathbb{F}_n^+ \quad \text{and } 1 \leq i \leq n,$$

where $c_i = \mu_{i,e}$.

Proof. For $w \in \mathbb{F}_n^+$, a computation shows that

$$T_i S_j \xi_w = (\mu_{j,w} \lambda_{i,wj}) \xi_{i w j}$$

and

$$S_j T_i \xi_w = (\lambda_{i,w} \mu_{j,iw}) \xi_{i w j}.$$

Thus the operators T_i and S_j commute precisely when

$$\mu_{j,iw} = \mu_{j,w} \lambda_{i,wj} \lambda_{i,w}^{-1}$$

for all w and $1 \leq i, j \leq n$. Using this formula repeatedly shows that

$$\mu_{j,iw} = \mu_{j,e} W(j, iw) W(e, iw)^{-1}.$$

This establishes equation (3) for $|w| \geq 1$, and it clearly holds for $w = e$. Notice also that an analogous equation for $\lambda_{i,w}$ in terms of $\mu_{i,w}$ can be derived. We use this fact to generate examples in Section 6. \blacksquare

Given a word $w = i_1 \cdots i_k$ in \mathbb{F}_n^+ , for operators S_i as defined in the lemma we put $S_w := S_{i_k} \cdots S_{i_1}$ (that the product is in reverse order is reflective of the connection with the right regular representation). We further define the weight function $W_\mu : \mathbb{F}_n^+ \times \mathbb{F}_n^+ \rightarrow \mathbb{R}_+$ by

$$(4) \quad W_\mu(v, w) = \mu_{i_1,v} \mu_{i_2,vi_1} \cdots \mu_{i_k,vi_1 \cdots i_{k-1}},$$

Observe that $S_w \xi_v = W_\mu(v, w) \xi_{vw}$ and $W_\mu(\cdot, \cdot)$ satisfies the cocycle condition

$$(5) \quad W_\mu(u, vw) = W_\mu(u, v) W_\mu(uv, w).$$

Once again, this can be seen through the direct computation in the following proof, or by viewing a weighted ‘right’ Fock space tree determined by the actions of S_1, \dots, S_n .

Proposition 2.2. *The weight function $W_\mu : \mathbb{F}_n^+ \times \mathbb{F}_n^+ \rightarrow \mathbb{R}_+$ satisfies the cocycle condition (5).*

Proof. Let $u, v = i_1 \cdots i_k$ and $w = j_1 \cdots j_l$ be words in \mathbb{F}_n^+ . Then by definition we have

$$\begin{aligned} W_\mu(u, vw) &= (\mu_{i_1,u} \cdots \mu_{i_k,ui_1 \cdots i_{k-1}}) (\mu_{j_1,uv} \cdots \mu_{j_l,uvj_1 \cdots j_{l-1}}) \\ &= W_\mu(u, v) W_\mu(uv, w). \end{aligned}$$

Lastly, the formula clearly holds when $v = e$ or $w = e$. \blacksquare

We can describe the commutant of those shifts which satisfy the condition, specific to the non-commutative multi-variable setting, which appears in Lemma 2.1. We note that for the rest of this section, our approach mirrors that of Section 1 from the Davidson and Pitts paper [5], where the commutant of \mathfrak{L}_n was computed. We wish to minimize redundancy, hence we shall leave details to the reader when part of a proof follows the lines of [5]. Instead we focus on the new aspects here.

Theorem 2.3. *Suppose the weights $\Lambda = \{\lambda_{i,w}\}$ associated with $T = (T_1, \dots, T_n)$ satisfy*

$$(6) \quad \sup_{i,w} W(i, w)W(e, w)^{-1} < \infty.$$

For $1 \leq i \leq n$, let $S_i \in \mathcal{B}(\mathcal{H}_n)$ be defined by

$$S_i \xi_w = \mu_{i,w} \xi_{wi} \quad \text{where} \quad \mu_{i,w} = W(i, w)W(e, w)^{-1}.$$

Let \mathfrak{R}_Λ be the unital WOT-closed algebra generated by $\{S_1, \dots, S_n\}$. Then the commutant of \mathfrak{R}_Λ coincides with \mathfrak{L}_Λ .

Proof. From the lemma we have \mathfrak{L}_Λ contained in the commutant \mathfrak{R}'_Λ . To establish the converse inclusion fix $A \in \mathfrak{R}'_\Lambda$ and set $A\xi_e = \sum_w a_w \xi_w$. Consider the operators

$$p_k(A) = \sum_{|w| < k} \left(1 - \frac{|w|}{k}\right) a_w W(e, w)^{-1} T_w,$$

which clearly belong to \mathfrak{L}_Λ , and hence to \mathfrak{R}'_Λ as well.

For $k \geq 0$, let Q_k denote the projection onto $\text{span}\{\xi_w : |w| = k\}$. For $X \in \mathcal{B}(\mathcal{H}_n)$, let $\Phi_j(X) = \sum_{k \geq \max\{0, -j\}} Q_k X Q_{k+j}$ and put

$$\Sigma_k(X) = \sum_{|j| < k} \left(1 - \frac{|j|}{k}\right) \Phi_j(X) \quad \text{for} \quad k \geq 1.$$

As observed in [5], the generalized Cesaro sums $\Sigma_k(X)$ converge to X in the strong operator topology. But observe in this case that $\Phi_j(A) \in \mathfrak{R}'_\Lambda$, since $S_i Q_k = Q_{k+1} S_i$ and hence $S_i \Phi_j(A) = \Phi_j(A) S_i$. Thus the sequence $\Sigma_k(A)$ belongs to \mathfrak{R}'_Λ and converges SOT to A .

However, we also have

$$\Sigma_k(A) \xi_e = \sum_{|w| < k} \left(1 - \frac{|w|}{k}\right) a_w \xi_w = p_k(A) \xi_e.$$

Thus for $w \in \mathbb{F}_n^+$ it follows that

$$\begin{aligned}\Sigma_k(A)\xi_w &= W_\mu(e, w)^{-1} S_w \Sigma_k(A) \xi_e \\ &= W_\mu(e, w)^{-1} S_w p_k(A) \xi_e = p_k(A) \xi_w.\end{aligned}$$

Hence A belongs to \mathfrak{L}_Λ , and therefore $\mathfrak{L}_\Lambda = \mathfrak{R}'_\Lambda$ as claimed. \blacksquare

Remark 2.4. Notice that for $n = 1$, condition (6) is simply the requirement that the weights be bounded above, hence satisfied for all shifts. Further, in this case the $\mu_{i,w}$ are just a constant multiple of the weights for the shift. It follows that Theorem 2.3 generalizes the commutant theorems discussed at the start of the section for the unweighted $n \geq 2$ case [5, 18], and for standard weighted shifts [21, 22]. While the class of shifts satisfying (6) for $n \geq 2$ is large, it is not all-encompassing. We provide examples in Section 6.

Corollary 2.5. *If Λ satisfies (6), then $\mathfrak{L}'_\Lambda = \mathfrak{R}_\Lambda$.*

Proof. This is not as trivial as the unweighted case, which follows by symmetry. Nonetheless, it is true for the shifts satisfying (6) since the associated S_i will be bounded weighted right creation operators commuting with the T_i , and hence the previous proof can be followed along with the roles of S_i and T_j reversed. \blacksquare

There are a number of other consequences of the commutant theorem. The proofs are simple and essentially the same as the unweighted case [5], hence we leave the details to the interested reader.

Corollary 2.6. *For Λ satisfying condition (6) we have:*

- (i) \mathfrak{L}_Λ is its own double commutant, $\mathfrak{L}''_\Lambda = \mathfrak{L}_\Lambda$.
- (ii) \mathfrak{L}_Λ is inverse closed.
- (iii) The only normal elements in \mathfrak{L}_Λ are the scalars.

Before continuing we point out a helpful computational lemma, which shows that elements of \mathfrak{L}_Λ have a generalized Fourier expansion. We remark that for $n = 1$, these Fourier expansions can be used to define weighted H^∞ and H^2 spaces determined by the weight sequence of a shift [7, 8, 21]. The key advantage of this approach is that many problems can then be phrased in terms of function theory. Some of this theory clearly goes through for $n \geq 2$, but we do not require this machinery here.

Lemma 2.7. *Suppose Λ satisfies (6). Let $A \in \mathfrak{L}_\Lambda$ and put $A\xi_e = \sum_w a_w \xi_w$. Then*

$$(7) \quad A\xi_v = W_\mu(e, v)^{-1} \sum_w a_w W_\mu(w, v) \xi_{wv} \quad \text{for } v \in \mathbb{F}_n^+.$$

Proof. From Theorem 2.3 we have

$$A\xi_v = W_\mu(e, v)^{-1} S_v A\xi_e = W_\mu(e, v)^{-1} \sum_w a_w W_\mu(w, v) \xi_{wv}. \quad \blacksquare$$

For the rest of this section we assume that all $\Lambda = \{\lambda_{i,w}\}$ we consider satisfy condition (6).

Proposition 2.8. *Every non-zero $A \in \mathfrak{L}_\Lambda$ is injective.*

Proof. For non-zero A in \mathfrak{L}_Λ there is a word v with $A\xi_v \neq 0$, thus by equation (7) we see $A\xi_e \neq 0$. Let v_1 be a word of minimal length for which $a_{v_1} \neq 0$. Suppose $A\xi = 0$ with $\xi = \sum_w b_w \xi_w \neq 0$, and choose a word v_2 of minimal length such that $b_{v_2} \neq 0$. Then again from Lemma 2.7, and by the minimality of v_1 and v_2 , we have

$$\begin{aligned} 0 = (A\xi, \xi_{v_1 v_2}) &= \sum_{w,u} b_w a_u W_\mu(e, w)^{-1} W_\mu(u, w) (\xi_{uw}, \xi_{v_1 v_2}) \\ &= a_{v_1} b_{v_2} W_\mu(e, v_2)^{-1} W_\mu(v_1, v_2) \neq 0. \end{aligned}$$

This contradiction shows that A is injective. \blacksquare

Since every non-trivial idempotent has kernel we have the following.

Corollary 2.9. *The algebra \mathfrak{L}_Λ contains no non-trivial idempotents.*

With an extra technical condition on weights we can prove \mathfrak{L}_Λ is semisimple.

Theorem 2.10. *If the weights $\Lambda = \{\lambda_{i,w}\}$ satisfy*

$$(8) \quad c_\Lambda = \inf_{v \in \mathbb{F}_n^+} \left(W_\mu(e, v)^{-1} \liminf_{k \geq 0} [W_\mu(v, v^{k-1})]^{1/k} \right) > 0,$$

then every non-zero $A \in \mathfrak{L}_\Lambda$ has non-zero spectrum.

Proof. As before, given non-zero $A \in \mathfrak{L}_\Lambda$ with $A\xi_e = \sum_w a_w \xi_w$ we let v be a word of minimal length such that $a_v \neq 0$. By using equation (7)

repeatedly, and the cocycle equation for W_μ , we find

$$\begin{aligned}
A^k \xi_e &= \sum_{w_1, \dots, w_k \in \mathbb{F}_n^+} a_{w_1} \cdots a_{w_k} [W_\mu(e, w_1) W_\mu(e, w_2 w_1) \cdots \\
&\quad W_\mu(e, w_{k-1} \cdots w_1)]^{-1} [W_\mu(w_2, w_1) W_\mu(w_3, w_2 w_1) \cdots \\
&\quad W_\mu(w_k, w_{k-1} \cdots w_1)] \xi_{w_k \cdots w_1} \\
&= \sum_{w_1, \dots, w_k} a_{w_1} \cdots a_{w_k} [W_\mu(e, w_1) \cdots W_\mu(e, w_{k-1})]^{-1} \\
&\quad W_\mu(w_k, w_{k-1} \cdots w_1) \xi_{w_k \cdots w_1} \\
&= (a_v^k W_\mu(e, v)^{-k} W_\mu(v, v^{k-1})) \xi_{v^k} + \sum_{w \neq v^k, |w| \geq |v|^k} b_w \xi_w.
\end{aligned}$$

The second equality follows from the identity

$$W_\mu(e, w_j \cdots w_1) = W_\mu(e, w_j) W_\mu(w_j, w_{j-1} \cdots w_1),$$

for $1 \leq j < k$, which is a special case of (5). Thus condition (8) gives us an $\varepsilon > 0$ such that the inequality

$$\|A^k\|^{1/k} \geq |(A^k \xi_e, \xi_{v^k})|^{1/k} \geq |a_v|(c_\Lambda - \varepsilon) > 0,$$

holds for all sufficiently large k . Therefore, it follows that A has positive spectral radius. \blacksquare

This result appears to be new for $n = 1$. The conclusion of the theorem is the same for \mathfrak{L}_n [5], hence together with Proposition 2.8 the following consequence is proved in the same way. In Section 6 we present examples which satisfy both (6) and (8).

Corollary 2.11. *If Λ satisfies (6) and (8), then the algebra \mathfrak{L}_Λ contains no quasinilpotent elements. It follows that \mathfrak{L}_Λ is semisimple, and the spectrum of every non-scalar element in \mathfrak{L}_Λ is connected with more than one point.*

3. REFLEXIVITY OF \mathfrak{L}_Λ

Recall that given an operator algebra \mathfrak{A} and a collection of subspaces \mathcal{L} , the subspace lattice $\text{Lat } \mathfrak{A}$ consists of those subspaces left invariant by every member of \mathfrak{A} , and the algebra $\text{Alg } \mathcal{L}$ consists of all operators which leave every subspace in \mathcal{L} invariant. Every algebra satisfies $\mathfrak{A} \subseteq \text{Alg Lat } \mathfrak{A}$, and an algebra \mathfrak{A} is *reflexive* if $\mathfrak{A} = \text{Alg Lat } \mathfrak{A}$.

We begin with the main result of this section. But first observe that we can regard the eigenvalues of \mathfrak{L}_Λ^* as forming a subset of \mathbb{C}^n , since every eigenvector ξ for the adjoint algebra satisfies equations $T_i^* \xi = \bar{\lambda}_i \xi$

for $1 \leq i \leq n$. Given $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and a word $w = i_1 \cdots i_k \in \mathbb{F}_n^+$, we shall write $w(\lambda) = \lambda_{i_1} \cdots \lambda_{i_k} \in \mathbb{C}$.

Theorem 3.1. *Suppose Λ satisfies condition (6). If the set of eigenvalues for \mathfrak{L}_Λ^* contains an open set \mathbb{U} in \mathbb{C}^n , then \mathfrak{L}_Λ is reflexive.*

Proof. Let $A \in \text{Alg Lat } \mathfrak{L}_\Lambda$. Let ν_λ be an eigenvector satisfying $T_i^* \nu_\lambda = \bar{\lambda}_i \nu_\lambda$ for $1 \leq i \leq n$. Since $\{\nu_\lambda\}^\perp$ belongs to $\text{Lat } \mathfrak{L}_\Lambda \subseteq \text{Lat } A$, we have $A^* \nu_\lambda = \bar{\alpha}_\lambda \nu_\lambda$ for some scalars α_λ and all $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_n) \in \mathbb{U}$. Let $A\xi_e = \sum_w a_w \xi_w$, then a computation yields

$$\begin{aligned} \alpha_\lambda(\xi_e, \nu_\lambda) = (A\xi_e, \nu_\lambda) &= \sum_w a_w W(e, w)^{-1}(\xi_e, T_w^* \nu_\lambda) \\ &= \sum_w a_w W(e, w)^{-1} w(\lambda)(\xi_e, \nu_\lambda). \end{aligned}$$

But $(\xi_e, \nu_\lambda) \neq 0$ for all $\bar{\lambda} \in \mathbb{U}$. There are a number of ways to see this, including an explicit formula for ν_λ which we derive in the next section. Thus $\alpha_\lambda = \sum_w a_w W(e, w)^{-1} w(\lambda)$ for $\bar{\lambda} \in \mathbb{U}$.

Let $v \in \mathbb{F}_n^+$, and notice that the subspace $\text{span}\{\xi_{wv} : w \in \mathbb{F}_n^+\}$ belongs to $\text{Lat } \mathfrak{L}_\Lambda$, hence is invariant for A . Thus we may write $A\xi_v = \sum_w b_w \xi_{wv}$ for some scalars b_w , which gives

$$\begin{aligned} (A\xi_v, \nu_\lambda) &= \sum_w b_w W(e, wv)^{-1}(T_{wv}^* \xi_e, \nu_\lambda) \\ &= \sum_w b_w [W(e, v)W(v, w)]^{-1} w(\lambda)v(\lambda)(\xi_e, \nu_\lambda). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (A\xi_v, \nu_\lambda) &= W(e, v)^{-1}(\xi_e, T_v^* A^* \nu_\lambda) \\ &= W(e, v)^{-1} \alpha_\lambda v(\lambda)(\xi_e, \nu_\lambda). \end{aligned}$$

It follows that $\alpha_\lambda = \sum_w b_w W(v, w)^{-1} w(\lambda)$ for $\bar{\lambda} \in \mathbb{U}$. We shall see in Theorem 4.3 and its corollaries, that the hereditary nature of the eigenvalue set for \mathfrak{L}_Λ^* allows us to assume that the origin in \mathbb{C}^n belongs to \mathbb{U} . Thus, as \mathbb{U} is an open set containing the origin, and hence contains a polydisc about the origin, the theory of analytic functions of several variables [20] gives us $b_w W(v, w)^{-1} = a_w W(e, w)^{-1}$ for $w \in \mathbb{F}_n^+$. In particular,

$$A\xi_v = \sum_w a_w W(v, w)W(e, w)^{-1} \xi_{wv}.$$

Recall from Theorem 2.3 that $\mathfrak{L}_\Lambda = \mathfrak{R}'_\Lambda$, and \mathfrak{R}_Λ is generated by S_1, \dots, S_n (and the unit). Using the definition of S_i we get:

$$(AS_i)\xi_v = \mu_{i,v}A\xi_{vi} = \sum_w a_w (\mu_{i,v}W(vi, w)W(e, w)^{-1}) \xi_{wvi}$$

and

$$(S_iA)\xi_v = \sum_w a_w (W(v, w)W(e, w)^{-1}\mu_{i,wv}) \xi_{wvi}.$$

But from the definition of $\mu_{i,w}$, and the cocycle equation (2) for $W(\cdot, \cdot)$, we see that

$$\begin{aligned} \mu_{i,v}W(vi, w) &= W(i, v)W(e, v)^{-1}W(vi, w) \\ &= W(e, v)^{-1}W(i, wv) \\ &= W(e, wv)^{-1}W(v, w)W(i, wv) \\ &= \mu_{i,wv}W(v, w) \end{aligned}$$

Heuristically, both sides of this equation consist of the product of weights from i to wvi in the weighted Fock space tree, divided by the product of weights from e to v . Therefore, $AS_i = S_iA$ for $1 \leq i \leq n$, and A belongs to $\mathfrak{R}'_\Lambda = \mathfrak{L}_\Lambda$, as required. \blacksquare

Remark 3.2. This gives a new elementary proof of the reflexivity of \mathfrak{L}_n for $n \geq 2$. Indeed, in [1] reflexivity is proved using non-commutative factorization results for \mathfrak{L}_n , and the authors point out that their proof does not carry over to the commutative case. Further, in [5] the stronger notion of hyper-reflexivity is proved for \mathfrak{L}_n , hence subsuming reflexivity. But as the authors point out, it is not clear how just reflexivity follows from their proof without using the stronger notion, whose proof uses some deep facts from von Neumann algebra theory. In the event, our proof simply relies on several variable analytic function theory in a way which carries over to the commutative case $n = 1$. We also mention that the paper [12] on *free semigroupoid algebras* contains a different elementary proof of reflexivity for \mathfrak{L}_n .

The hypothesis of the theorem can be weakened to simply require an eigenvalue $\lambda = (\lambda_1, \dots, \lambda_n)$ for \mathfrak{L}_Λ^* with each $\lambda_i \neq 0$. For if this is the case, Corollary 4.5 shows there is also an open set of eigenvalues about the origin.

For $n = 1$, every weighted shift T for which T^* has a non-zero eigenvalue is reflexive. In other words, the algebra generated by T is reflexive [21]. However, when there is a non-zero eigenvalue for T^* , the spectral theory for unilateral weighted shifts implies there is an entire disc of such eigenvalues. Thus Theorem 3.1 is a generalization of this result.

By using Corollary 4.7, we obtain examples which fulfil the hypothesis of the theorem in Section 6.

At the present time we have no examples of non-reflexive algebras \mathfrak{L}_Λ for $n \geq 2$. There are numerous examples in the single variable case; for instance, every unicellular weighted shift operator [9, 16, 21] generates a non-reflexive algebra. This motivates the following problem.

Problem 3.3. Is there an analogue of unicellular weighted shifts in the non-commutative multivariable setting?

For $v \in \mathbb{F}_n^+$, let E_v be the subspace $E_v = \text{span}\{\xi_{wv} : w \in \mathbb{F}_n^+\}$. If $T = (T_1, \dots, T_n)$ is a weighted shift, then each E_v is clearly invariant for each T_i , and thus belongs to $\text{Lat } \mathfrak{L}_\Lambda$. A natural guess for a generalization of unicellular weighted shifts here would be $\text{Lat } \mathfrak{L}_\Lambda$ equal to the subspace lattice \mathcal{L} generated by $\{E_v : v \in \mathbb{F}_n^+\}$. But there are simple examples which show this cannot happen for $n \geq 2$. Indeed, the subspace $\mathcal{M} = \text{span}\{\xi_e, \frac{1}{\sqrt{2}}(\xi_1 + \xi_2)\}$ clearly belongs to $\text{Lat } \mathfrak{L}_\Lambda^*$ for all Λ , and hence \mathcal{M}^\perp is a subspace outside of \mathcal{L} which belongs to every $\text{Lat } \mathfrak{L}_\Lambda$.

More generally, it would also be interesting to characterize when \mathfrak{L}_Λ is reflexive. This appears to be an open problem for $n = 1$ still.

Problem 3.4. Determine which algebras \mathfrak{L}_Λ are reflexive.

4. EIGENVALUES FOR \mathfrak{L}_Λ^*

It is clear that unilateral weighted shifts have no non-zero eigenvalues, and it is also obvious that the T_i from a shift $T = (T_1, \dots, T_n)$ have no non-zero eigenvalues. However, the adjoint of a unilateral weighted shift has a wealth of non-zero eigenvalues if it has any which are non-zero. We mention that for $n = 1$ the point spectrum for the adjoint of a weighted shift is either an open disc, possibly with some boundary points included, or just the origin [10, 21]. For $n \geq 2$, the eigenvalues for \mathfrak{L}_n^* form the open complex unit n -ball $\mathbb{B}_n = \{\lambda \in \mathbb{C}^n : \|\lambda\|_2 < 1\}$ [1, 5]. In the eigenvalue analysis of this section we discover geometry not present in either of these settings. Let us begin the discussion by observing some basic facts.

Proposition 4.1. *If $T = (T_1, \dots, T_n)$ is a weighted shift, then $T_i^* \xi_e = 0$ for $1 \leq i \leq n$, and hence $\vec{0} \in \mathbb{C}^n$ is always an eigenvalue for \mathfrak{L}_Λ^* . In fact more is true,*

$$\bigcap_{i=1}^n \ker T_i^* = \text{span}\{\xi_e\}.$$

Proof. It is clear that $T_i^* = W_i L_i^*$ annihilates the vacuum vector. If ξ satisfies $T_i^* \xi = W_i L_i^* \xi = 0$, then $L_i^* \xi = 0$ since W_i is injective. But the projection $P_e = \xi_e \xi_e^*$ onto the vacuum space is given by the equation $P_e = I - \sum_{i=1}^n L_i L_i^*$. Hence $P_e \xi = \xi$ and ξ belongs to $\text{span}\{\xi_e\}$. ■

More generally, if $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and some unit vector ξ in \mathcal{H}_n satisfy $T_i^* \xi = \bar{\lambda}_i \xi$, then

$$\|\lambda\|_2^2 = \sum_{i=1}^n |\lambda_i|^2 = \sum_{i=1}^n (T_i T_i^* \xi, \xi) = (T T^* \xi, \xi) \leq \|T\|^2 = \sup_{i,w} \lambda_{i,w}^2.$$

Thus all eigenvalues for \mathfrak{L}_Λ^* are contained in the corresponding ball in \mathbb{C}^n determined by the supremum of $\Lambda = \{\lambda_{i,w}\}$. In general this estimate is not a good one though.

We now determine the form of all joint eigenvectors. Notice we make no extra assumptions on weights in the next three results.

Lemma 4.2. *Let $T = (T_1, \dots, T_n)$ be a weighted shift with weights Λ . If a vector $\xi = \sum_w \bar{a}_w \xi_w \in \mathcal{H}_n$ satisfies $T_i^* \xi = \bar{\lambda}_i \xi$ for $1 \leq i \leq n$, then*

$$(9) \quad a_w = a_e w(\lambda) W(e, w)^{-1} \quad \text{for } w \in \mathbb{F}_n^+.$$

Conversely, if ξ belongs to \mathcal{H}_n with coefficients $\overline{(\xi, \xi_w)}$ satisfying (9), normalized so that $a_e = (\xi, \xi_e) = 1$, then $T_i^ \xi = \bar{\lambda}_i \xi$ for $1 \leq i \leq n$. It follows that every eigenspace for \mathfrak{L}_Λ^* is one-dimensional.*

Proof. If $\xi = \sum_w \bar{a}_w \xi_w$ is in \mathcal{H}_n with $T_i^* \xi = \bar{\lambda}_i \xi$, then

$$\overline{\lambda_i a_e} = (\bar{\lambda}_i \xi, \xi_e) = (T_i^* \xi, \xi_e) = \lambda_{i,e} (\xi, \xi_i) = \bar{a}_i \lambda_{i,e},$$

so that $a_i = a_e \lambda_i \lambda_{i,e}^{-1}$. Another computation gives the next level:

$$\overline{\lambda_i a_j} = (\bar{\lambda}_i \xi, \xi_j) = (T_i^* \xi, \xi_j) = \lambda_{i,j} (\xi, \xi_{ij}) = \bar{a}_{ij} \lambda_{i,j},$$

hence $a_{ij} = a_j \lambda_i \lambda_{i,j}^{-1} = a_e (\lambda_i \lambda_j) (\lambda_{i,j} \lambda_{j,e})^{-1}$. We can now obtain equation (9) by induction.

On the other hand, if $\xi = \sum_w \overline{w(\lambda)} W(e, w)^{-1} \xi_w$ belongs to \mathcal{H}_n , then since $T_i^* = W_i L_i^*$ we have

$$\begin{aligned} T_i^* \xi = W_i L_i^* \xi &= \sum_{w=iu; u \in \mathbb{F}_n^+} \overline{w(\lambda)} W(e, w)^{-1} W_i L_i^* \xi_w \\ &= \sum_u \overline{(iu)(\lambda)} W(e, iu)^{-1} \lambda_{i,u} \xi_u = \bar{\lambda}_i \xi. \end{aligned}$$

It also follows that for every eigenvalue $\lambda = (\lambda_1, \dots, \lambda_n)$ of \mathfrak{L}_Λ^* , the eigenspace $E_\lambda = \text{span}\{\xi : T_i^* \xi = \bar{\lambda}_i \xi \text{ for } 1 \leq i \leq n\}$ is one-dimensional. ■

As an immediate consequence we get a tight characterization of the eigenvalues for \mathfrak{L}_Λ^* which generalizes the known result for $n = 1$ [8].

Theorem 4.3. *The eigenvalues for \mathfrak{L}_Λ^* consist of all $\lambda \in \mathbb{C}^n$ for which*

$$(10) \quad \sum_{w \in \mathbb{F}_n^+} |w(\lambda)|^2 W(e, w)^{-2} < \infty.$$

In particular, it is evident that the set of eigenvalues has the following hereditary property.

Corollary 4.4. *If $\mu = (\mu_1, \dots, \mu_n)$ is an eigenvalue for \mathfrak{L}_Λ^* , then so is every element of the set*

$$\mathbb{D} = \{(\lambda_1, \dots, \lambda_n) : |\lambda_i| \leq |\mu_i| \text{ for } 1 \leq i \leq n\}.$$

Proof. If λ belongs to \mathbb{D} , then the series $\sum_w |w(\lambda)|^2 W(e, w)^{-2}$ converges since its partial sums are bounded above by the corresponding series for μ , and the result follows from the theorem. \blacksquare

As a special case of this result, together with Theorem 3.1, we discover a method for finding reflexive algebras.

Corollary 4.5. *If Λ satisfies condition (6), and if \mathfrak{L}_Λ^* has an eigenvalue $\mu = (\mu_1, \dots, \mu_n)$ with each $\mu_i \neq 0$, then the eigenvalues for \mathfrak{L}_Λ^* include a polydisc in \mathbb{C}^n , and hence \mathfrak{L}_Λ is reflexive.*

Let us be more concrete. Specifically, when the weights $\Lambda = \{\lambda_{i,w}\}$ for a shift are bounded away from zero we identify ellipses as eigenvalue sets.

Corollary 4.6. *If $\Lambda = \{\lambda_{i,w}\}$ is bounded away from zero, then the eigenvalues for \mathfrak{L}_Λ^* include the set*

$$\mathbb{E} = \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n \mid \frac{|\lambda_1|^2}{c_1^2} + \dots + \frac{|\lambda_n|^2}{c_n^2} < 1 \right\},$$

where $c_i = \inf_{w \in \mathbb{F}_n^+} \lambda_{i,w} > 0$ for $1 \leq i \leq n$.

Proof. Let $c = (c_1, \dots, c_n)$. Then for $w = i_k \cdots i_1 \in \mathbb{F}_n^+$ we have

$$w(c) = c_{i_k} \cdots c_{i_1} \leq \lambda_{i_k, i_{k-1} \cdots i_1} \cdots \lambda_{i_2, i_1} \lambda_{i_1, e} = W(e, w).$$

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ belong to \mathbb{E} and put $\lambda c^{-1} = (\lambda_1 c_1^{-1}, \dots, \lambda_n c_n^{-1})$. Then $\|\lambda c^{-1}\|_2 < 1$, and hence

$$\begin{aligned} \sum_{w \in \mathbb{F}_n^+} |w(\lambda)|^2 W(e, w)^{-2} &= \sum_w |w(\lambda c^{-1})|^2 w(c)^2 W(e, w)^{-2} \\ &\leq \sum_w |w(\lambda c^{-1})|^2 \\ &= \sum_{k \geq 0} \left(\sum_{i=1}^n |\lambda_i|^2 c_i^{-2} \right)^k \\ &= (1 - \|\lambda c^{-1}\|_2^2)^{-1} < \infty. \end{aligned}$$

Thus by Theorem 4.3, every $\lambda = (\lambda_1, \dots, \lambda_n)$ in \mathbb{E} is an eigenvalue for \mathfrak{L}_Λ^* . \blacksquare

Therefore, with Theorem 3.1 this result gives us a large subclass of reflexive algebras. We give examples of such shifts in Section 6.

Corollary 4.7. *If $\Lambda = \{\lambda_{i,w}\}$ is bounded away from zero and satisfies (6), then \mathfrak{L}_Λ is reflexive.*

5. JOINT SPECTRAL THEORY

Our analysis in the previous section gives us motivation for discussing the joint spectral theory for non-commuting n -tuples of operators. We mention the seminal work of Taylor [23, 24] on multi-variable spectral theory. As suggested at the end of the paper [23], a natural definition for a (non-commuting) n -tuple of operators $A = (A_1, \dots, A_n)$ on \mathcal{H} to be non-singular would be to require the existence of another n -tuple $B = (B_1, \dots, B_n)$ for which

$$(11) \quad A_1 B_1 + \dots + A_n B_n = I$$

and

$$(12) \quad B_i A_j = \delta_{ij} I \quad \text{for } 1 \leq i, j \leq n.$$

This is a natural notion since it says the map $A : \mathcal{H}^{(n)} \rightarrow \mathcal{H}$ given by $A[x_1 \cdots x_n]^t = \sum_{i=1}^n A_i x_i$ is invertible. Indeed, equations (11) and (12) can be written as $AB^t = I_{\mathcal{H}}$ and $B^t A = I_{\mathcal{H}^{(n)}}$.

Hence we shall use the following nomenclature: the *joint right spectrum* $\sigma_r(A)$ of $A = (A_1, \dots, A_n)$ consists of $\lambda \in \mathbb{C}^n$ for which no solution $B = (B_1, \dots, B_n)$ exists to the equation

$$\sum_{i=1}^n (A_i - \lambda_i I) B_i = I.$$

Whereas the *joint left spectrum* $\sigma_l(A)$ consists of $\lambda \in \mathbb{C}^n$ for which no solution $B = (B_1, \dots, B_n)$ exists to the equations

$$B_i(A_j - \lambda_j I) = \delta_{ij} I \quad \text{for } 1 \leq i, j \leq n.$$

The *full spectrum* $\sigma(A)$ is the union of $\sigma_r(A)$ and $\sigma_l(A)$.

Our goal in this section is to show that, while the right spectrum can yield good information, the above notions of left and full spectra are inappropriate for the non-commutative multi-variable setting. We accomplish this by focusing on the unweighted case. Let us begin by computing the right spectrum of the left creation operators.

Theorem 5.1. *The right spectrum of $L = (L_1, \dots, L_n)$ is $\sigma_r(L) = \overline{\mathbb{B}}_n$.*

Proof. From Corollary 4.6, it follows that every $\lambda \in \mathbb{B}_n$ determines an eigenvector $\nu_\lambda = \sum_w \overline{w(\lambda)} \xi_w$ for \mathfrak{L}_n^* (The eigenvalues for the unweighted case \mathfrak{L}_n^* were initially worked out in [1, 5]). Thus for $\lambda \in \mathbb{B}_n$ there can be no solution $B = (B_1, \dots, B_n)$ to $\sum_{i=1}^n B_i^*(L_i^* - \overline{\lambda_i} I) = I$, so that $\mathbb{B}_n \subseteq \sigma_r(L)$, and a standard approximation argument shows that $\overline{\mathbb{B}}_n \subseteq \sigma_r(L)$.

On the other hand, for $\lambda \in \mathbb{B}_n$ the operator $I - \sum_{i=1}^n \overline{\lambda_i} L_i$ is invertible since

$$\left\| \sum_i \overline{\lambda_i} L_i \right\|^2 = \sum_i |\lambda_i|^2 = \|\lambda\|^2 < 1.$$

A computation shows that $(I - \sum_{i=1}^n \overline{\lambda_i} L_i)^{-1} = \sum_w \overline{w(\lambda)} L_w$ as linear transformations, hence the latter operator is bounded and the sum is a WOT-limit. Thus if $\lambda \in \mathbb{C}^n$ with $\|\lambda\| > 1$, and defining $\lambda^{-1} = \frac{1}{\|\lambda\|^2} (\overline{\lambda_1}, \dots, \overline{\lambda_n})$, we have $\sum_w \overline{w(\lambda^{-1})} L_w$ belonging to the inverse closed algebra \mathfrak{L}_n . For $1 \leq i \leq n$, define

$$B_i = \frac{\overline{\lambda_i}}{\|\lambda\|^2} \sum_w \overline{w(\lambda^{-1})} L_w.$$

It follows that

$$\sum_{i=1}^n (\lambda_i I - L_i) B_i = \sum_w \overline{w(\lambda^{-1})} L_w - \sum_{i=1}^n \left(\sum_{u=iw, w \in \mathbb{F}_n^+} \overline{u(\lambda^{-1})} L_u \right) = I,$$

and hence $\lambda \in \sigma_r(L)^c$. Therefore, $\sigma_r(L) = \overline{\mathbb{B}}_n$ as claimed. \blacksquare

This theorem generalizes the well-known result for $n = 1$, that the right spectrum of the unilateral weighted shift operator is the closed unit disc. Of course, in the single variable case we know this is the entire spectrum. The rigidity in the definition of left invertibility for $n \geq 2$, results in $L = (L_1, \dots, L_n)$ having a very large left spectrum.

Proposition 5.2. *Let $\mathbb{D}_n = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : |\lambda_i| < 1\}$ be the unit polydisc in \mathbb{C}^n . The left spectrum of $L = (L_1, \dots, L_n)$ includes the complement of this set,*

$$\mathbb{C}^n \setminus \mathbb{D}_n \subseteq \sigma_l(L).$$

Proof. For the sake of brevity, we focus on the $n = 2$ case. We are required to show there are no solutions $A = (A_1, A_2)$ to

$$(13) \quad A_i(L_j - \lambda_j I) = \delta_{ij} I \quad \text{for } 1 \leq i, j \leq 2,$$

when $|\lambda_1| \geq 1$ or $|\lambda_2| \geq 1$. Suppose there was such a solution. Then $A_2 L_1 = \lambda_1 A_2$, and if $|\lambda_1| > 1$ we would have

$$A_2 \xi_{1^k} = \lambda_1 A_2 \xi_{1^{k-1}} = \dots = \lambda_1^k A_2 \xi_e \quad \text{for } k \geq 1$$

implying the unboundedness of A_2 unless $A_2 \xi_e = 0$, which is addressed below.

If instead $|\lambda_1| = 1$, then $A_1 L_1 = \lambda_1 A_1 + I$, and for $k \geq 1$ we would have

$$\begin{aligned} A_1 \xi_{1^k} &= A_1 L_1 \xi_{1^{k-1}} = \lambda_1 A_1 \xi_{1^{k-1}} + \xi_{1^{k-1}} \\ &\vdots \\ &= \lambda_1^k A_1 \xi_e + \lambda_1^{k-1} \xi_e + \lambda_1^{k-2} \xi_1 + \dots + \xi_{1^{k-1}}. \end{aligned}$$

Hence $\lim_{k \rightarrow \infty} \|A_1 \xi_{1^k}\| = \infty$, implying the unboundedness of A_1 . Similarly, $|\lambda_2| = 1$ implies that A_2 is unbounded. Finally, if $|\lambda_2| > 1$ and $A_2 \xi_e = 0$, then the equation $A_2 L_2 = \lambda_2 A_2 + I$ can be applied in the same manner to obtain $\lim_{k \rightarrow \infty} \|A_2 \xi_{2^k}\| = \infty$. Thus, in all cases A_1 or A_2 would have to be unbounded. \blacksquare

Note 5.3. Notice there can be solutions to (13) for certain values of $\lambda = (\lambda_1, \lambda_2)$. For instance, the solutions $A = (A_1, A_2)$ for $\lambda = \vec{0}$ are rank one perturbations of (L_1^*, L_2^*) of the form

$$A_i = L_i^* + \eta_i \xi_e^* \quad \text{for } \eta_i \in \mathcal{H}_2.$$

Indeed, these are solutions to (13) for $\lambda = \vec{0}$ since (L_1^*, L_2^*) is and

$$(\eta_i \xi_e^*) L_j = \eta_i (L_j^* \xi_e)^* = 0 \quad \text{for } 1 \leq i, j \leq 2.$$

Conversely, if $A = (A_1, A_2)$ is a solution for $\lambda = \vec{0}$, then

$$\begin{aligned} A_1 &= A_1 L_1 L_1^* + A_1 L_2 L_2^* + A_1 P_e \\ &= L_1^* + (A_1 \xi_e) \xi_e^*, \end{aligned}$$

and the analogue is true for A_2 .

It should be possible to say more about the left spectrum of L . In fact, we believe $\sigma_l(L)$ includes $\mathbb{C}^n \setminus \overline{\mathbb{B}}_n$, in other words that there are *no*

solutions to (13) for $||\lambda_2|| > 1$. This would imply that the full spectrum of L is all of \mathbb{C}^n . Nonetheless, we can say the following.

Corollary 5.4. *The full spectrum of $L = (L_1, \dots, L_n)$ includes the set*

$$\overline{\mathbb{B}}_n \cup (\mathbb{C}^n \setminus \mathbb{D}_n) \subseteq \sigma(L).$$

Remark 5.5. Thus, towards a joint spectral theory for non-commuting n -tuples of operators, it seems unreasonable to consider the definition for the full spectrum discussed at the start of the section. Especially considering the fact that $L = (L_1, \dots, L_n)$ is fundamental to the theory of non-commuting n -tuples of operators, playing the role of the unilateral shift in this setting. However, we have seen that the right spectrum can yield good information for an n -tuple. We mention another possibility for the left spectrum might be to use the equation $BA^t = \sum_{i=1}^n B_i A_i = I$ to define left invertibility. For instance, it appears that with this definition it may be possible to generalize the result from [19] for the left spectra of weighted shift operators to this non-commutative setting.

6. EXAMPLES

In this section we describe examples of shifts satisfying the various weight conditions discovered in the paper. We begin by presenting a simple subclass satisfying (6).

Example 6.1. The set of shifts satisfying (6) is clearly closed under changing finitely many weights. For instance, the shifts

$$T = (c_1 L_1, \dots, c_n L_n), \quad \text{for } c_i > 0$$

are easily seen to satisfy (6). In fact since the T_i are just multiples of L_i , the algebra they generate is \mathfrak{L}_n . Let $\lambda_{i,w} > 0$ for $1 \leq i \leq n$ and $|w| \leq k$, and put $\lambda_{i,w} = c_i$ for $|w| > k$. Then the shift \tilde{T} determined by $\Lambda = \{\lambda_{i,w}\}$ will satisfy (6). For, if $|w| > k$ with $w = i_s \cdots i_1$, then

$$\lambda_{i_j, i_{j-1} \cdots i_1} = c_{i_j} = \lambda_{i_j, (i_{j-1} \cdots i_1)i} \quad \text{for } k < j \leq s \quad \text{and } 1 \leq i \leq n.$$

Thus the weight products $W(e, w)$ and $W(i, w)$ will be determined by the original n -tuple when we move far enough down the tree.

Observe that the operators which determine the n -tuple \tilde{T} will be particular finite rank perturbations of the $c_i L_i$ in this case. Further, the weights are bounded away from zero in this class, hence by Corollary 4.7 the associated algebras \mathfrak{L}_Λ are reflexive.

We next show that condition (6) is not satisfied by all shifts.

Example 6.2. For simplicity consider $n = 2$. Let $T = (T_1, T_2)$ be a weighted shift which satisfies

$$\lambda_{1,1^k} = m^{-1} \quad \text{and} \quad \lambda_{1,1^k 2} = m^{-1/2} \quad \text{for } k \geq 0.$$

Then $W(e, 1^k) = m^{-k}$ and $W(2, 1^k) = m^{-k/2}$, so that

$$\sup_{i,w} W(i, w) W(e, w)^{-1} \geq \sup_{k \geq 0} W(2, 1^k) W(e, 1^k)^{-1} = \lim_{k \rightarrow \infty} m^{k/2}.$$

Thus if $m > 1$, equation (6) is not satisfied.

We can also use this example to find shifts which do satisfy (6). Indeed, consider the above weights with $m \leq 1$, and define all other weights $\lambda_{i,w} \equiv c$ for some constant $c \leq \sqrt{m}$. If $w = 1^k$, we have $W(1, w) = m^{-k}$, so that

$$W(i, 1^k) W(e, 1^k)^{-1} \leq \max\{1, m^{k/2}\} \leq 1$$

for $i = 1, 2$, by the previous paragraph. Here are the other cases for a word w of length k :

$$W(e, w) = \begin{cases} m^{-s} c^{k-s} & \text{if } w = u 2 1^s, \text{ with } s > 0 \\ m^{-s/2} c^{k-s} & \text{if } w = 1^s 2 \text{ or } w = u 2 1^s 2, \text{ with } s \geq 0 \end{cases}$$

Whereas, for $i = 1, 2$ we have

$$W(i, w) = \begin{cases} m^{-s} c^{k-s} & \text{if } w = u 2 1^s, \text{ with } s > 0 \text{ and } i = 1 \\ m^{-s/2} c^{k-s} & \text{if } w = u 2 1^s, \text{ with } s > 0 \text{ and } i = 2 \\ c^k & \text{if } w = 1^s 2 \text{ or } w = u 2 1^s 2, \text{ with } s \geq 0 \end{cases}$$

This exhausts all cases, hence it follows that the supremum in (6) is equal to 1. Further, the weights in this class of examples are bounded away from zero, hence by Corollary 4.7 the associated algebras \mathfrak{L}_Λ are reflexive.

The periodic weighted shifts introduced in [13] provide a useful subclass for the purposes here.

Example 6.3. For $k \geq 1$, a weighted shift $T = (T_1, \dots, T_n)$ is of *period* k if for all $w \in \mathbb{F}_n^+$ we have

$$T_i \xi_w = \lambda_{i,u} \xi_{iw},$$

where $w = uv$ is the unique decomposition of w with $0 \leq |u| < k$ and $|v| \equiv 0 \pmod{k}$. The *remainder* scalars $\{\lambda_{i,u} : 0 \leq |u| < k\}$ completely determine the shift. It is most satisfying to think of this notion of periodicity in terms of the weighted Fock space tree, which is generated by the finite weighted tree top with vertices $\{\xi_w : |w| \leq k\}$ and weighted edges given by the remainder scalars. In [13], this notion of periodicity was used to find non-commutative generalizations of the Bunce-Deddens C^* -algebras.

Notice the 1-periodic shifts are simply of the form $(c_1 L_1, \dots, c_n L_n)$, but for $k \geq 2$ the k -periodic shifts give us a large non-trivial class to work with. For the sake of brevity, we focus on the 2-periodic shifts $T = (T_1, T_2)$; that is, we consider $k = 2$ and $n = 2$. Each such 2-tuple will be determined by six scalars, which we denote by

$$\lambda_{1,e} = a, \lambda_{2,e} = b, \lambda_{1,1} = c, \lambda_{2,1} = d, \lambda_{1,2} = e, \lambda_{2,2} = f.$$

We first observe that periodic shifts can fail to satisfy (6). For $k \geq 1$, consider the words $w_k = (21)^k \in \mathbb{F}_2^+$. Then by 2-periodicity we have

$$W(e, w_k) = (ad)^k \quad \text{and} \quad W(2, w_k) = (eb)^k.$$

Consequently, if $eb > ad > 0$, then

$$\sup_{i,w} W(i, w) W(e, w)^{-1} \geq \sup_{k \geq 0} W(2, w_k) W(e, w_k)^{-1} = \infty,$$

showing that (6) fails in this case. Nevertheless, there are many periodic shifts which do satisfy (6).

The set Λ associated with every periodic shift with non-zero weights is plainly bounded away from zero. Hence by Corollary 4.7, the algebra \mathfrak{L}_Λ generated by a periodic shift which satisfies (6) is reflexive. As a simple subclass of examples, consider the case when $a = b > 0$ and $c = d = e = f > 0$. In this case we compute

$$W(e, w) = \begin{cases} a^k c^k & \text{if } |w| = 2k \\ a^{k+1} c^k & \text{if } |w| = 2k + 1 \end{cases}$$

Whereas, for $i = 1, 2$ we have

$$W(i, w) = \begin{cases} c^k a^k & \text{if } |w| = 2k \\ c^{k+1} a^k & \text{if } |w| = 2k + 1 \end{cases}$$

Thus (6) is clearly satisfied for the entire subclass, with ca^{-1} providing an upper bound, and hence all of the associated algebras \mathfrak{L}_Λ are reflexive.

The next observation follows directly from the analysis in Section 2, and will allow us to generate non-trivial examples satisfying (8).

Proposition 6.4. *Let $S = (S_1, \dots, S_n)$ be operators in $\mathcal{B}(\mathcal{H}_n)$ defined by $S_i \xi_w = \mu_{i,w} \xi_{wi}$ for scalars $\mu_{i,w} > 0$, normalized so that each $\mu_{i,e} = 1$. Let $\widetilde{W} : \mathbb{F}_n^+ \times \mathbb{F}_n^+ \rightarrow \mathbb{R}_+$ be the weight function defined by $\widetilde{W}(v, e) = 1$ for all $v \in \mathbb{F}_n^+$ and*

$$\widetilde{W}(v, w) = \mu_{i_1,v} \mu_{i_2,vi_1} \cdots \mu_{i_k,vi_1 \cdots i_{k-1}} \quad \text{if } w = i_1 \cdots i_k.$$

Suppose that

$$(14) \quad \sup_{i,w} \widetilde{W}(e, w)^{-1} \widetilde{W}(i, w) < \infty.$$

Let $T_i \in \mathcal{B}(\mathcal{H}_n)$ be defined by $T_i \xi_w = \lambda_{i,w} \xi_{iw}$ where

$$\lambda_{i,w} = \widetilde{W}(i, w) \widetilde{W}(e, w)^{-1}.$$

Then $T = (T_1, \dots, T_n)$ satisfies (6), and $S = (S_1, \dots, S_n)$ are the weighted right creation operators obtained for T as in Theorem 2.3. In particular, the functions $\widetilde{W} = W_\mu$ are the same.

Proof. By following the proof of Lemma 2.1, we see that the S_i and T_j commute, and we have the corresponding formulas for $\lambda_{i,w}$ and $\mu_{i,w}$ in terms of the other. The n -tuple $T = (T_1, \dots, T_n)$ satisfies (6) precisely because the $\mu_{i,w}$ are uniformly bounded. Thus the S_i really are the weighted right creation operators obtained in Theorem 2.3 which generate the commutant, and hence $\widetilde{W} = W_\mu$. ■

Example 6.5. Let $S = (S_1, \dots, S_n)$ be weighted right creation operators with non-zero weights $\mu_{i,w}$ for which there is a $k \geq 0$ with $\mu_{i,w} \equiv 1$ for $|w| > k$ (There is complete freedom on weight choices for $|w| \leq k$). Since only finitely many weights are different than 1, it is easy to see that $W_\mu = \widetilde{W}$ satisfies (8) and (14).

Thus by Proposition 6.4, the corresponding weighted left creation operators $T = (T_1, \dots, T_n)$ satisfy (6), and hence the associated algebras \mathfrak{L}_Λ are semisimple. We note that, while \mathfrak{L}_Λ has this structure, the weights $\Lambda = \{\lambda_{i,w}\}$ are not easily described. Indeed, an examination of the formula $\lambda_{i,w} = W_\mu(i, w) W_\mu(e, w)^{-1}$ shows that typically these scalars will not satisfy a finiteness condition analogous to the one which defines $\mu_{i,w}$. In other words, the S_i will be finite rank perturbations of the R_i , but the T_i will not in general be finite rank perturbations of the L_i .

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